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SOLUTION OF STATIONARY HEAT-CONDUCTION PROBLEMS FOR CURVILINEAR REGIONS USING EIGENFUNCTION EXPANSIONS (FUNDAMENTAL SOLUTIONS)

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Formulas for solution of stationary problems of heat conduction in bodies of a curvilinear shape have been obtained in explicit form using eigenfunction expansions; an analogous solution has been constructed for temperature fluctuations. An algorithm of computation of the boundary functions for classical regions has been proposed; these functions make it possible to reduce the boundary conditions of the problem to a homogeneous form. The exact fundamental solutions in the region of a rectangle with arbitrary smooth boundary conditions of the 1st kind have been constructed using them. These solutions are fundamental, since they can be used when boundary-value problems and inverse problems with unknown boundary conditions are considered for a wide range of curvilinear regions.

Keywords: *heat-conduction equation, curvilinear regions, eigenfunctions, exact solutions.*

Introduction. We know of different methods of solution of stationary heat-conduction problems for curvilinear regions: the methods of separation of variables, of double- and single-layer potentials [1, 2], of Green's functions, and of integral transformations, the Ritz and Bubnov–Galerkin methods [1–4], finite-difference methods [5] and those of the theory of functions of complex variable [6], the angular-superposition method [7], and many others. In our opinion, the method of spectral expansions of the sought functions in generalized Fourier series, which is proposed below, is more promising, since the generalized Fourier series converge uniformly and quite rapidly [8]. The error is dependent only on the accuracy of computations of eigenfunctions and on the number of terms used in the Fourier series. For classical regions, the eigenfunctions and eigenvalues are known exactly [1, 2, 4]; for curvilinear regions, they can be found by the angular-superposition method [7, 9].

Stationary Temperature Field. Let it be necessary to find, in a certain simply connected curvilinear region Ω with piecewise smooth boundary Γ, the solution of the heat-conduction problem [1, 2]

$$\Delta T = -\tilde{q}(x)\,, \quad L_\Gamma(T)|_\Gamma = \varphi(x_\Gamma)\,, \quad x \in \Omega\,, \quad x_\Gamma \in \Gamma\,, \tag{1}$$

where $\tilde{q}$ is the heat sources. Below, the multidimensional coordinates of the Euclidean space $\{x\}$ will be denoted by x in general considerations, and the Cartesian coordinates will be denoted by (x, y) in partial problems. In finding the solution, we use eigenfunction theory [8]; therefore, we reduce the boundary conditions to a homogeneous form, representing the unknown function by the sum

$$T(x) = M(x) + V(x)\,, \quad L_\Gamma(M)|_\Gamma = \varphi(x_\Gamma)\,, \quad L_\Gamma(V)|_\Gamma = 0\,, \quad M \in L_p^{\alpha+2}(\Omega)\,, \quad V \in L_p^{\alpha}(\Omega)\,. \tag{2}$$

The problem of construction of $M(x)$ consists of the boundary conditions and the smoothness conditions indicated in (2); then we obtain, for the unknown $V(x)$, the problem with homogeneous conditions

$$\Delta V = -q(x)\,, \quad L_\Gamma(V)|_\Gamma = 0\,, \quad x \in \Omega\,, \quad x_\Gamma \in \Gamma\,, \quad q(x) = \tilde{q}(x) + \Delta M \in L_p^{\alpha}(\Omega)\,. \tag{3}$$

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We associate problem (3) with then auxiliary problem on finding the full range (spectrum) of eigenfunctions and eigenvalues $\{G_i\}$ and $\{\lambda_i\}$:

$$\Delta G_i + \lambda_i G_i = 0\,, \quad L_\Gamma (G_i)|_\Gamma = 0\,, \quad x \in \Omega\,, \quad x_1 \in \Gamma\,. \tag{4}$$

Problems (3) and (4) are formulated for the same region Ω with boundary Γ and equal homogeneous boundary conditions. We will assume that the ranges of eigenfunctions and eigenvalues $\{G_i\}$ and $\{\lambda_i\}$ are known from solution of problem (4); therefore, we can expand $V(x)$ and the right-hand side $q(x)$ of the differential equation (3) in Fourier series [8]

$$V = \sum_{i=1}^{\infty} v_i G_i\,, \quad q = \sum_{i=1}^{\infty} q_i G_i\,, \quad (v_i, q_i) = \frac{1}{N_i} \iint_\Omega (V, q)\, G_i ds\,, \quad N_i = \iint_\Omega G_i^2 ds\,. \tag{5}$$

Such an approach makes it possible to reduce the problem on finding the unknown function to determination of the unknown expansion coefficients v_i. To this end, we multiply the Poisson equation (3) by G_i and integrate over the region Ω:

$$\iint_\Omega \Delta V G_i ds = -\iint_\Omega q G_i ds\,. \tag{6}$$

The left-hand side of (6) will be transformed with allowance for the second Green formula [2] and for expression (4):

$$\iint_\Omega \Delta V G_i ds = \iint_\Omega V \Delta G_i ds = -\lambda_i \iint_\Omega V G_i ds = -\lambda_i v_i N_i\,. \tag{7}$$

Using (7) and (5) we represent Eq. (6) in the form

$$\lambda_i v_i = q_i \Rightarrow v_i = q_i / \lambda_i\,. \tag{8}$$

Substituting v_i from (8) into (5) and (2), we find the sought solution of problem (1)

$$T = M + \sum_{i=1}^{\infty} \frac{q_i}{\lambda_i} G_i\,. \tag{9}$$

To use formula (9) we must construct the boundary function M satisfying conditions from (2), determine the ranges of eigenfunctions and eigenvalues $\{G_i\}$ and $\{\lambda_i\}$ by solution of problem (4), compute the Fourier coefficients q_i using (5), and substitute all the quantities into (9).

Algorithm of Construction of the Boundary Functions in General Form. The boundary function $M(x)$ is determined not uniquely and is found in practice by selection or from any auxiliary considerations, so that it satisfies the inhomogeneous boundary conditions (2). Construction of $M(x)$ for the one-dimensional region Ω presents no special problems [2, 3], but it is quite difficult in the case of multidimensional problems. If Ω has a smooth curvilinear boundary, to this end, we use the angular-superposition method ([10]). Below, we consider certain cases of classical regions with two coordinates.

In constructing $M(x, y)$, we will use the following definition of the concept of matching of the boundary conditions on sides in the general case of a curvilinear polygon: additional equalities between the functions prescribed in the boundary conditions will be called the matchings of these conditions. The indicated equalities are a consequence of the continuity of either $M(x, y)$ or its partial derivatives of a certain prescribed order at angular points of the region Ω, as these points are approached from different directions along the sides forming the angle in question.

We formulate the basic propositions of the algorithm in constructing the boundary functions.

1. All f_i functions from the boundary conditions on the sides of the polygon (i is the side number), which are prescribed on the ith smooth portion of the piecewise smooth boundary Γ, are set independent, despite the fact that in particular cases they can also be coincident as functions dependent on equal coordinates.

2. The boundary conditions in problems (1)–(3) must be continuous and matched so that $M \in C^{(2)}(\overline{\Omega})$. If they are not matched in a given problem, we should correct, by introduction of small ε vicinities near Γ or around singular points where we have a discontinuity (mismatch), the initial formulation of the boundary-value problem by changing $\tilde{q}$ in (1) or should change f_i so that these conditions are matched in the changed problem. Thereafter the parameter ε can be selected as small as desired depending on the required accuracy.

3. The function $M(x)$ will be represented as the sum of several terms, each linearly involving f_i from the boundary conditions specified on the sides of the curvilinear polygon. These terms appear as the product of the unknown function A_i of one variable along the normal to the corresponding boundary and the sum $(f_i + X_i)$ dependent on the variables along the boundary. In the case of two variables the terms will appear in two variants:

$$A_i(y)\,[f_i(x) + X_i(x)] \quad \text{or} \quad A_j(x)\,[f_j(y) + X_j(y)]\,, \tag{10}$$

the variables x and y are separated in each term.

4. The number of terms is equal to the number of boundary conditions on all sides of the curvilinear polygon.

5. The function with the smallest curvature should be selected among the set of possible expressions of $M(x)$, which improves the accuracy and diminishes the time of subsequent computer-aided calculations.

The proposed construction of $M(x, y)$ is convenient, since it is linearly dependent on f_i used in the boundary conditions in general form, which will make it possible to subsequently solve applied problems for multiply connected regions of an intricate shape.

Using this algorithm we construct the boundary functions for a rectangular region and a circular sector. The region of the rectangle will be determined by the inequalities

$$0 \le x \le a\,, \quad 0 \le y \le b\,, \quad (x, y) \in \overline{\Omega}\,.$$

We write the Dirichlet boundary conditions on the sides of the rectangle

$$T|_{\Gamma} = M|_{\Gamma} \Rightarrow M(x, y)|_{y=0} = f_1(x)\,, \quad M(x, y)|_{x=a} = f_2(y)\,,$$
$$M(x, y)|_{y=b} = f_3(x)\,, \quad M(x, y)|_{x=0} = f_4(y)\,, \quad f_i \in L_p^{\alpha+2}(\Omega)\,, \quad i = 1\text{–}4\,. \tag{11}$$

The matching conditions follow from the continuity of $M(x, y)$ at the angles of the rectangle, as the angles are approached from two directions along the sides of a given angle

$$f_1(0) = f_4(0)\,, \quad f_1(a) = f_2(0)\,, \quad f_2(b) = f_3(a)\,, \quad f_3(0) = f_4(b)\,. \tag{12}$$

In what follows, we will assume that conditions (12) are fulfilled. Here we have four boundaries with four conditions; therefore, we represent $M(x, y)$ by the sum of four terms of the (10) type with separated variables

$$M(x, y) = A_1(y)\,[f_1(x) + X_1(x)] + A_2(x)\,[f_2(y) + X_2(y)]$$
$$+ A_3(y)\,[f_3(x) + X_3(x)] + A_4(x)\,[f_4(y) + X_4(y)]\,, \tag{13}$$

where A_k, X_k (k = 1–4) are the functions of one variable indicated in (13) yet to be known. To find the unknown functions A_k and X_k we successively substitute $M(x, y)$ into boundary conditions (11):

$$A_1(0)\,[f_1(x) + X_1(x)] + A_2(x)\,[f_2(0) + X_2(0)]$$
$$+ A_3(0)\,[f_3(x) + X_3(x)] + A_4(x)\,[f_4(0) + X_4(0)] = f_1(x)\,,$$

$$A_1(y)[f_1(a) + X_1(a)] + A_2(a)[f_2(y) + X_2(y)]$$

$$+ A_3(y)[f_3(a) + X_3(a)] + A_4(a)[f_4(y) + X_4(y)] = f_2(y) ,$$

$$A_1(b)[f_1(x) + X_1(x)] + A_2(x)[f_2(b) + X_2(b)] \tag{14}$$

$$+ A_3(b)[f_3(x) + X_3(x)] + A_4(x)[f_4(b) + X_4(b)] = f_3(x) ,$$

$$A_1(y)[f_1(0) + X_1(0)] + A_2(0)[f_2(y) + X_2(y)]$$

$$+ A_3(y)[f_3(0) + X_3(0)] + A_4(0)[f_4(y) + X_4(y)] = f_4(y) .$$

The first equation of system (14) involves only the variables $f_1(x)$ and $f_3(x)$ prescribed independently; therefore, they may not be related by this equation. We apply analogous considerations to each equation from (14), which leads to equalities

$$A_1(0) = A_2(a) = A_3(b) = A_4(0) = 1 , \quad A_1(b) = A_2(0) = A_3(0) = A_4(a) = 0 . \tag{15}$$

From system (14), allowing for (15), we find X_k:

$$X_1(x) = - A_2(x)[f_2(0) + X_2(0)] - A_4(x)[f_4(0) + X_4(0)] ,$$

$$X_2(y) = - A_1(y)[f_1(a) + X_1(a)] - A_3(y)[f_3(a) + X_3(a)] , \tag{16}$$

$$X_3(x) = - A_2(x)[f_2(b) + X_2(b)] - A_4(x)[f_4(b) + X_4(b)] ,$$

$$X_4(y) = - A_1(y)[f_1(0) + X_1(0)] - A_3(y)[f_3(0) + X_3(0)] .$$

Setting successively $x = 0$, $x = a$, $y = 0$, and $y = b$ in (16), with allowance for (15) we obtain four auxiliary expressions

$$f_4(0) + X_1(0) + X_4(0) = 0 , \quad f_2(0) + X_1(a) + X_2(0) = 0 ,$$

$$f_3(a) + X_2(b) + X_3(a) = 0 , \quad f_4(b) + X_3(0) + X_4(b) = 0 . \tag{17}$$

The remaining four equalities from (16) will be coincident with (17) as a corollary of conditions (12) and (15). If we now replace X_k in (13) by their expressions (16) and use relations (17), we obtain the sought boundary function

$$M(x, y) = A_1(y) f_1(x) + A_2(x) f_2(y) + A_3(y) f_3(x) + A_4(x) f_4(y)$$

$$- A_1(y) A_2(x) f_1(a) - A_1(y) A_4(x) f_1(0) - A_2(x) A_3(y) f_2(b) - A_3(y) A_4(x) f_3(0) . \tag{18}$$

We can select as A_k any smooth functions satisfying boundary conditions (15) and convenient for spectral expansions to impart a specific form to the dependence (18). We propose two variants:

$$A_1(y) = \cos\frac{\pi y}{2b} , \quad A_2(x) = \sin\frac{\pi x}{2a} , \quad A_3(y) = \sin\frac{\pi y}{2b} , \quad A_4(x) = \cos\frac{\pi x}{2a};$$

$$A_1(y) = 1 - \frac{y}{b} , \quad A_2(x) = \frac{x}{a} , \quad A_3(y) = \frac{y}{b} , \quad A_4(x) = 1 - \frac{x}{a} . \tag{19}$$

Analogously we find the boundary functions in specifying the Neumann conditions or mixed boundary conditions. For the circular sector with Dirichlet boundary conditions in cylindrical coordinates

$$\overline{\Omega} = (0 \le r \le R,\ 0 \le \varphi \le \varphi_0)\ ,\quad M(r,\varphi)|_{\varphi=0} = f_1(r)\ ,\quad M|_{r=R} = f_2(\varphi)\ ,\quad M|_{\varphi=\varphi_0} = f_3(r)$$

when the matching conditions

$$f_1(0) = f_3(0)\ ,\quad f_1(R) = f_2(0)\ ,\quad f_3(R) = f_2(\varphi_0)$$

are fulfilled, the boundary function appears as

$$M(r,\varphi) = A_1(\varphi) f_1(r) + A_3(\varphi) f_3(r) + A_2(r)\,[f_2(\varphi) - A_1(\varphi) f_1(R) - A_3(\varphi) f_3(R)]\ ,$$

$$A_1(0) = A_2(R) = A_3(\varphi_0) = 1\ ,\quad A_1(\varphi_0) = A_3(0) = 0\ .$$

As the functions A_i (i = 1–3) we can use the dependences

$$A_1(\varphi) = 1 - \frac{\varphi}{\varphi_0}\ ,\quad A_3(\varphi) = \frac{\varphi}{\varphi_0}\ ,\quad A_2 = 1\ .$$

The proposed algorithm of construction of boundary functions is applicable to an arbitrary orthogonal coordinate system; also, it can be generalized for multidimensional simply connected regions with different linear boundary conditions: for a rectangular parallelepiped, a circular cone bounded by one or two spherical surfaces, etc.

Exact Solution of the Stationary Heat-Conduction Problem for a Rectangle with Arbitrary Smooth Boundary Conditions. Here we will construct the solution in explicit form, which can subsequently be used in considering a fairly wide range of problems for simply connected curvilinear regions with piecewise smooth boundaries. Formula (9) has been obtained, when the boundary conditions in general form were specified; therefore, this formula may be thought of as being fundamental, since it is equivalent, in a sense, to the solution with the theory of functions of complex variable for a circle. This approach has substantial advantages: it dispenses with the need for any conformal mapping; furthermore, it allows formulation of the problems in the simplest coordinate system — a rectangular Cartesian system. The solutions for other classical regions, when the exact expressions of the boundary functions $M(x)$ with specified boundary conditions in general form and the full ranges of eigenfunctions and eigenvalues $\{G_i\}$ and $\{\lambda_i\}$ are known, may also be thought of as being fundamental. For the rectangular region with boundary conditions of the 1st kind, the eigenfunctions and eigenvalues are found by solution of problem (4) [2] and appear as

$$G_{m,n} = \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b}\ ,\quad \lambda_{m,n} = \left(m\frac{\pi}{a}\right)^2 + \left(n\frac{\pi}{b}\right)^2\ ,\quad (m, n) = 1, 2, \dots\ . \tag{20}$$

If expression (18) is taken as $M(x, y)$ and the second variant from (19) is taken for $A_1 - A_4$, the right-hand side $q(x, y)$ of the differential equation (3) takes the form

$$q(x, y) = \tilde{q}(x, y) + \left(1 - \frac{y}{b}\right) f_1''(x) + \frac{x}{a} f_2''(y) + \frac{y}{b} f_3''(x) + \left(1 - \frac{x}{a}\right) f_4''(y)\ . \tag{21}$$

In computing the coefficients $q_{m,n}$ of the function $q(x, y)$, we need the integrals

$$\begin{gathered}\frac{2}{a}\int_0^a \sin m\pi \frac{x}{a}\,dx = \frac{2}{m\pi}\left[1 - (-1)^m\right],\quad \frac{2}{b}\int_0^b \sin n\pi \frac{y}{b}\,dy = \frac{2}{n\pi}\left[1 - (-1)^n\right],\\ \frac{2}{a}\int_0^a \frac{x}{a} \sin m\pi \frac{x}{a}\,dx = \frac{-2}{m\pi}(-1)^m\ ,\quad \frac{2}{b}\int_0^b \frac{y}{b} \sin n\pi \frac{y}{b}\,dy = \frac{-2}{n\pi}(-1)^n\ .\end{gathered} \tag{22}$$

The heat sources $\tilde{q}(x, y)$ and the second derivatives $f_1''-f_4''$, which are assumed to be continuous in $\overline{\Omega}$, according to conditions (3), will be represented by uniformly convergent Fourier series:

$$\tilde{q}(x, y) = \sum_{(m,n)=1}^{\infty} \tilde{q}_{m,n} \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b}, \quad \tilde{q}_{m,n} = \frac{4}{ab}\int_0^a \left[\int_0^b \tilde{q}(x, y) \sin n\pi \frac{y}{b} dy\right] \sin m\pi \frac{x}{a} dx,$$

$$\left\{f_1''(x), f_3''(x)\right\} = \sum_{m=1}^{\infty} \left\{f_{1,m}'', f_{3,m}''\right\} \sin m\pi \frac{x}{a}, \quad \left\{f_{1,m}'', f_{3,m}''\right\} = \frac{2}{a}\int_0^a \left\{f_1''(x), f_3''(x)\right\} \sin m\pi \frac{x}{a} dx, \tag{23}$$

$$\left\{f_2''(y), f_4''(y)\right\} = \sum_{n=1}^{\infty} \left\{f_{2,n}'', f_{4,n}''\right\} \sin n\pi \frac{y}{b}, \quad \left\{f_{2,n}'', f_{4,n}''\right\} = \frac{2}{b}\int_0^b \left\{f_2''(y), f_4''(y)\right\} \sin n\pi \frac{y}{b} dy.$$

Using (20), (22), and (23), from (21) we obtain

$$q(x, y) = \sum_{(m,n)=1}^{\infty} q_{m,n} \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b}, \quad q_{m,n} = \frac{4}{ab}\int_0^a \int_0^b q(x, y) \sin n\pi \frac{y}{b} dy \sin m\pi \frac{x}{a} dx$$

$$= \tilde{q}_{m,n} + \frac{2}{n\pi} f_{1,m}'' - \frac{2}{m\pi}(-1)^m f_{2,n}'' - \frac{2}{n\pi}(-1)^n f_{3,m}'' + \frac{2}{m\pi} f_{4,n}''. \tag{24}$$

Thus, for the region of the rectangle, the fundamental solution (9) of the Poisson equation (1) with boundary conditions (11) written in general form is represented by the following formula:

$$T(x, y) = M(x, y) + \sum_{(m,n)=1}^{\infty} \frac{q_{m,n}}{\lambda_{m,n}} \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b}. \tag{25}$$

In the equality (25), we should take $M(x, y)$ from (18) and the Fourier coefficients $q_{m,n}$ from (24), whereas $\tilde{q}_{m,n}$, $f_{1,m}''$, $f_{2,n}''$, $f_{3,m}''$, and $f_{4,n}''$ are computed by the integrals (23).

Conclusions. Formulas (9) and (25) obtained above have a simple form, which makes it possible to use them in solving, with a high degree of accuracy, both primal and inverse boundary-value problems for a fairly wide range of curvilinear simply connected regions with piecewise smooth boundaries.

NOTATION

$A_i(y)$, $A_j(x)$, $A_j(\varphi)$, and $A_2(r)$, dimensionless functions dependent on just one indicated coordinate; a and b, dimensions of a rectangle, cm; $C^{(2)}$, functional space of continuous doubly differentiable functions; ds, area element or volume element depending on the dimension of the region Ω; f_k, functions from the boundary conditions for a rectangle; $f_{1,m}-f_{4,m}$, q_i, q_{1i}, q_{2i}, $q_{m,n}$, $\tilde{q}_{m,n}$, v_i, v_{1i}, and v_{2i}, spectral-expansion coefficients; G_i, eigenfunctions; L_p^{α}, classes of Sobolev–Liouville functions; L_Γ, linear homogeneous boundary operator corresponding to the 1st, 2nd, or 3d boundary-value problem; $M(x)$, boundary function; N_i, norm of eigenfunctions; $\tilde{q}$, q, $\tilde{q}_1$, and $\tilde{q}_2$, right-hand sides of the differential equations; (r, φ), cylindrical coordinates; R, radius of a cylinder; T, temperature; V, v_1, and v_2, unknown functions; x, independent variables of the Euclidean space; X_i, unknown functions acting as a variation from f_i; x_Γ, coordinates of points at the boundary Γ; (x, y), Cartesian coordinates; Δ, Laplace operator; ε, small parameter; λ_i, eigenvalues, cm^{-2}; $\underline{\varphi}(x_\Gamma)$, function from the boundary conditions for temperature; φ_0, angle of opening of a circular sector; Ω, region; $\overline{\Omega}$, region with its boundary. Subscripts: p, power of the modulus of the integrand of the Sobolev–Liouville class; Γ, belonging to the operator or the quantity to the boundary of the region; α, order of the derivative function of the Sobolev–Liouville class.

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